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A Review of Mesh Free- Element Free Galerkin method, Recent Development and Application

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Abstract

Numerical simulation using computers has increasingly become a very important approach for solving problems in engineering and science. It plays a valuable role in providing tests and examinations for theories, offering insights to complex physics, and assisting in the interpretation and even the discovery of new phenomena. FEM has been one of major tools in dealing with many engineering and academic problems from many decades; however it suffers from the problem of its heavy reliability on quality mesh which is not suitable when treating with large deformation problems. The term mesh less (or mesh free) method refers to a broad class of effective numerical techniques for solving a growing number of science and engineering applications without the dependence of an underlying computational mesh. The variety of problems analyzed by these methods is very large, and ranges from fracture mechanics, over fluid mechanics, multiscale problems, and laminated composites, all the way to moving material interfaces. The review presented in this paper mainly focuses various mesh free methods, their applications and requirements as well as advantages gained by coupling element free methods with the classical FEM. Sometimes it is even more beneficial to couple two mesh free methods to get better results. There are various methods to couple such mesh free methods like EFG, MLPG and SPH with FEM like master slave coupling, coupling via interface shape functions bridging domain coupling compatibility coupling with Lagrange multipliers and hybrid coupling.

Keywords- *FEM, MLPG, EFG, MESHFREE METHODS, LAMINATED COMPOSITES*

I. INTRODUCTION

Mesh free methods can become a good alternative to classical Finite Element Method and Finite Differential method as it has been proved more flexible and the possibility to use for very large deformation also. Mesh free methods seems to be more flexible for problems like large deformation, fracture and fragmentation, because they do not rely on a fixed topological connectivity between nodes. The main advantage of mesh free methods reveals on the fact while dealing with crack growth problems is that particle can be added or removed anywhere without the remesh which is necessary in the classical FEM method. As there is no need of remesh in mesh free methods adaptive refinement of discretization can be done easily. [1, 110]

In building a modern and advanced engineering system, engineers must undertake a very sophisticated process in modeling, simulation, visualization, analysis, designing, prototyping, testing, fabrication, and construction. The finite element methods are well established and powerful computational / simulation techniques which are used for modeling and analysis of physical phenomena in different fields of engineering and applied sciences. [7, 8,]It has successfully been applied for a large number of engineering applications, for example solid mechanics, structure mechanics, electro magnetism, geo-mechanics, bio mechanics, aerodynamics and so on, The closed examination of above difficulties associated with FEM reveals that the root cause is the heavy and rigid reliance on the use of quality elements that are the building blocks of FEM**.** The Finite Element Method (FEM) has been the standard tool for this kind of calculations, but it is also having some shortcomings which are mentioned below:

1. When dealing with large deformation problems, as elements in mesh becomes extremely skewed or compressed there will be considerable loss in accuracy of results.
2. Such distorted elements will not coincide the original mesh lines, which creates difficulties during modeling with such elements.
3. Consumes major time of analyst using FEM packages. It becomes a major component of the cost of a simulation project. The concern is more the manpower time, and less the computer time.
4. As FEM is essentially based on continuum mechanics, in which the elements formulated cannot be broken it is very difficult to simulate the breakage of material into a large number of fragments.
5. Post processing techniques are required to achieve smooth stress distribution in structural problems because of Discontinuous secondary variables like stresses across element boundaries. [1,9]

To overcome the above difficulties mesh free methods have been developed which do not require a mesh to discretize the problem and any connectivity between nodes. Mesh free methods are having following advantages as compared to finite element method**.**

1. There is no need to provide in advance any information about the relationship of the nodes, so it provides flexibility in adding and deleting nodes whenever and wherever needed.
2. It saves a lot of human effort as nodes are created by fully automated manner with the help of computers.
3. The solution so created by mesh-free methods is entirely in terms of set of nodes.
4. As connectivity can change in run time in mesh free methods, it can be suitable for very large deformation also. [1,9]

In spite of having many advantages it is to be questionable that why mesh free methods are not commercialized till now as compared to classical grid based method like FEM. this is because mesh free methods are in developing stage and the following drawbacks are still to be un- resolved.

1. Mesh free shape functions require highly accurate integration scheme to apply because mesh free

shape functions are the rationale functions unlike the grid base shape functions.

2. As shape functions of mesh free methods are not interpolants but approximants, the treatment of essential boundary conditions is not straight forward as in traditional mesh based methods. [9,110]

The finite element method for the modeling of complex problems in applied mechanics and related fields are well established, but the reliance on the method of mesh leads to complications for certain classes of problems. The basic difference between FEM and mesh free methods can easily be understood by following procedural chart [1, 110] :

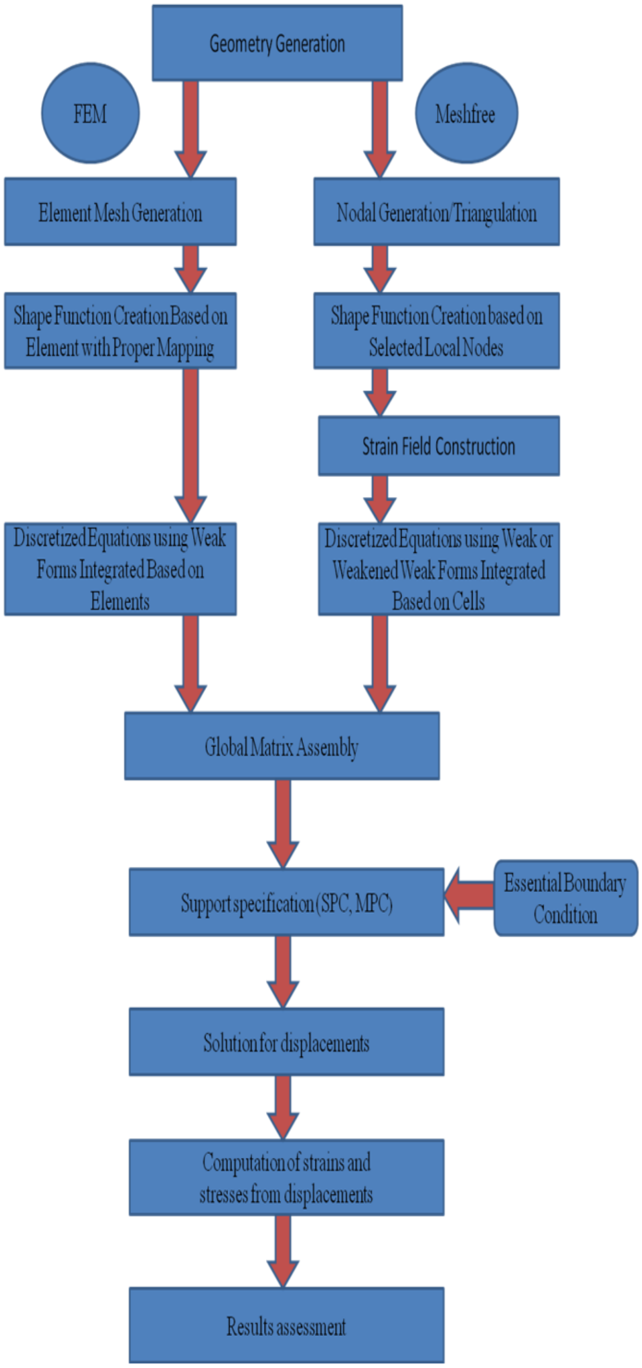

Figure 1.1 Basic Procedural difference in Grid based and Mesh free method:

There have been a number of mesh free methods developed so far to overcome the difficulties encountered in grid based methods. The common procedural steps followed by almost all such mesh less methods are briefed in the following chart [9, 110]:

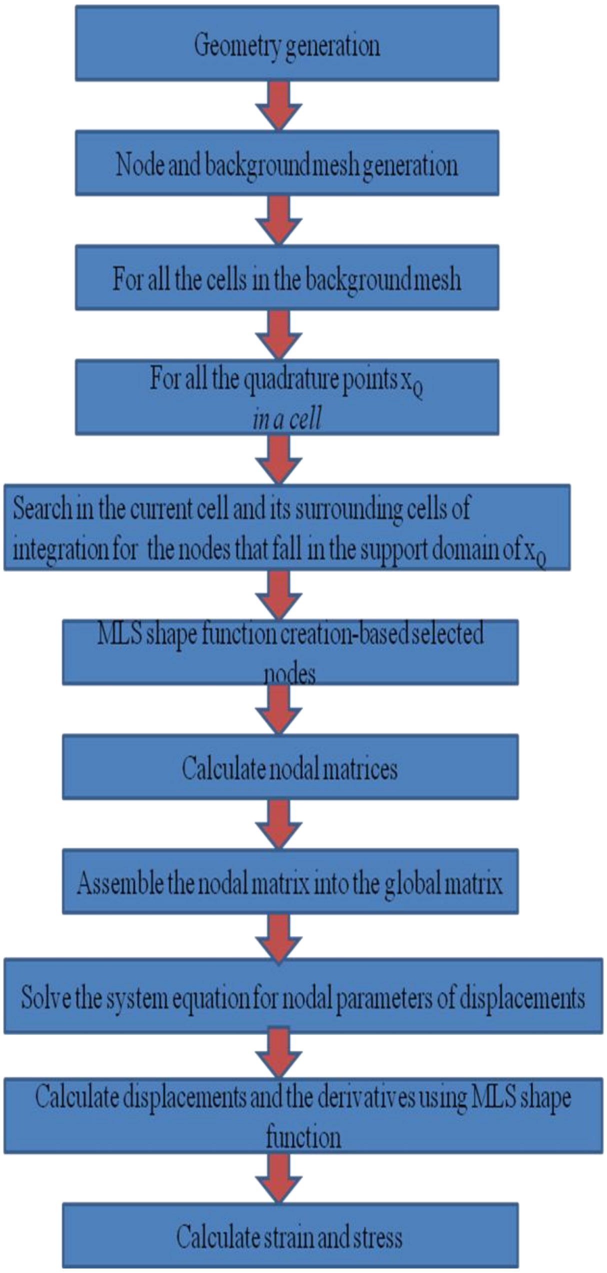

Figure 1.2 Basic procedural steps

1.1 Classification of Mesh free methods:

A recent strong interest is focused on the development of the next generation of computational methods — mesh free methods, which are expected to be superior to the conventional grid-based FDM and FEM for many applications. The key idea of the mesh free methods is to provide accurate and stable numerical solutions for integral equations or PDEs with all kinds of possible boundary conditions with a set of arbitrarily distributed nodes (or particles) without using any mesh that provides the connectivity of these nodes or particles. These meshfree methods with their approximation techniques are listed in the below table [1,19,20]:

Table 1.1

METHODS	REFERENCES	Methods of approximation
Smoothed Particle Hydrodynamics(SPH)	Lucy and Monaghan 1977 etc.	Integration representation
Finite Point Method	Liszka and Orkisz, 1980; onate et al.,1996,etc.	Finite difference representation
Diffuse Element Method (DEM)	Nayroles et al. , 1992	Moving Least Square (MLS) approximation and Galerkin method
Element Free Galerkin (EFG) Method	Belytschko et al. , 1994, 1996, 1998	MLS approximation and Galerkin method
Reproduce Kernel Particle Method(RKPM)	W. K. Liu et al., 1995-97	Integral representation and Galerkin method
HP-Cloud Method	Duarte and Oden, 1996	MLS approximation, Partition of Unity (PU)
Free Mesh Method	Yagawa and Yamada,1996;1998,etc.	Galerkin method
Meshless Local Petrov Galerkin(MLPG) Method	Atluri and Zhu 1998-99	MLS approximation, Petrov-Galerkin method
Point Interpolation Method(PIM)	G. R. Liu et al. 1999-2001	Point interpolation (radial and polynomial basis), Petro-Galerkin method
Meshfree Weak-Strong Form (MWS)	Liu and Gu in 2002-2003	MLS, PIM, radial PIM, Collocation and Petrov-Galerkin

II. Mesh free methods coupled with other methods:

All domain type and boundary type mesh free methods are having their own advantages and drawbacks. Recently the attempts have also been made to couple such methods to have the advantages of both methods. There is an additional motivation to couple meshfree methods that are formulated using moving least squares (MLS) shape functions and meshfree methods that are formulated using point interpolation method (PIM) shape functions or finite element (FE) shape functions. The objective of such coupling is to simplify the basic procedure of imposing essential boundary conditions. A number of combined methods have been formulated such as: EFG/BEM, EFG/HBEM, MLPG/FEM/BEM, EFG/FEM [44,89,94]

2.1 Element Free Galerkin (EFG) Method:

This method is based on moving least square approximations (MLS), initially presented by Lancaster and Salkauskas. Element Free galerkin method was first employed by Nyroles et al. for the development of Diffuse Element method. Belytschko et al. proposed Element Free Galerkin method by modifying and refining the DEM method. In the EFG method, the problem domain is discretized by a set of nodes scattered in the problem domain and on the boundaries of the domain. The MLS approximation procedure is then used to approximate the displacement field at a point of interest within the problem domain using the nodal parameters of displacement at the nodes in the support domain of the point. The approximate solution obteied then is used to have displacement and stress analysis. The major features of EFG are summerised as:

- Moving least square method is used to create shape functions
- Galerkin Weak Form creates discretized equations.
- A background mesh is created to carry out integration to obtain the system matrices.

The mesh is solely used for the purpose of integration which is completely independent of the number of field nodes or its density. [1,109,110,]

2.2 Formulation of EFG:

In this section the generalized formulation of Element Free Galerkin method has been presented with MLS approximation. In meshless approximation the approximation function can be expressed as, [1,21,29,51,108,110]

$$u^h(x,\bar{x}) = \sum_{i=1}^{m} P_i\,(\bar{x})\,a_i(x) \tag{1}$$

Where P_i (x¯) are monomial basis functions of order m,and a_i (x) are non-constant coefficients that are functions of spatial coordinates x. The coefficients a_i (x) are computed by performing a minimization of the difference between local approximation and the nodal values.

$$J = \sum_{I} w(x - x_I)\,[u^h(x, x_I) - u(x_I)]^2 \tag{2}$$

Where w(x-x_I) is called weight function With the solution of a_i (x), the final MLS approximation can be obtained as

$$u^h(x) = \sum_{I=1}^{n} \Phi_I(x) u_I \tag{3}$$

Where Φ_I (x) is the shape function, and n is the number of local nodes.

The nodal discrete equations are obtained using the constrained galerkin weak form using lagrange multiplier and then the governing matrices for boundary condition, stiffness, force and displacement can be obtained. Such matrices for two dimensional problems are listed below:

$$K_{IJ} = \int_{\Omega} B_I^T D B_J d\Omega \tag{4}$$

$$G_{IK} = -\int_{\Gamma_u} \Phi_i N_K \, d\Gamma \tag{5}$$

$$f_I = \int_{\Gamma_t} \Phi_I \, t d\Gamma + \int_{\Omega} \Phi_I b d\Omega \tag{6}$$

$$q_K = -\int_{\Gamma_u} N_K \, u d\Gamma \tag{7}$$

Where,

$$B_I = \begin{bmatrix} \Phi_{I,x} & 0 \\ 0 & \Phi_{I,y} \\ \Phi_{I,y} & \Phi_{I,x} \end{bmatrix} \tag{8}$$

$$N_K = \begin{bmatrix} N_K & 0 \\ 0 & N_K \end{bmatrix} \tag{9}$$

$$N_K = \begin{bmatrix} N_K & 0 \\ 0 & N_K \end{bmatrix} \tag{10}$$

These relations are for plane stresses. In which a comma designates a partial derivative with respect to the indicated spatial variable; E and ν are Young's modulus and Poison's ratio respectively. The above discrete nodal equations are assembled into global matrix: [1,2,65]

$$\begin{bmatrix} K & G \\ G^T & 0 \end{bmatrix} \begin{Bmatrix} u \\ \lambda \end{Bmatrix} = \begin{Bmatrix} f \\ q \end{Bmatrix} \tag{11}$$

The essential boundary condition has been imposed and the global matrix is solved to obtain nodal displacement parameters.

2.3 EFG- Application:

There are numerous improvements has been witnessed since the appearance of Element Free Galerkin method. The application of EFG has spanned various areas of Engineering and has established as dependable solution technique. The various areas of application of EFG covers crack problems, static and free vibration, buckling and post buckling, non- linear analysis and transient dynamics of structure: [

2.3.1 Crack Problems:

The most earlier application of EFG is in crack growth problems. The EFG method with linear MLS approximations was used to study the two-dimensional elastostatic and elastodynamic fracture problems [3,4,10,11,14,16,] , such as crack growth from a fillet, crack propagation in concrete, and edge-cracked plate under impact loading. Krysl and Belytschko [23]conducted the modelling of arbitrary three-dimensional dynamically propagating cracks in elastic bodies using EFG with explicit time integration. The several examples like simulation of mixed-mode growth of centre through crack in a finite plate, mode-1 surface breaking penny shaped crack in a cube, penny shaped crack growing under general mixed-mode conditions, tortion-tension rectangular bar with centre-through crack. It is found that, compared to FEM, the EFG method is more suitable for crack problems because it does not require remeshing and avoids the need for excessive refinement near the crack front. A boundary element free method, a variant of the EFG method, was employed for fracture analysis of 2D piezoelectric solids and the interaction between collinear interfacial cracks. Some crack problems were also dealt with by using modified EFG method [37,38,48,58,63,87,]. Recently, Sun et al. [88] presented a meshfree simulation of cracking and failure of structures by combining the EFG method and a cohesive segment method, and Zhang et al. [90] investigated the 2D fracture problems via an improved element-free Galerkin method. A boundary element free method, a variant of the EFG method, was employed for fracture analysis of 2D piezoelectric solids [77] and the interaction between collinear interfacial cracks [86]

2.3.2Static Analysis of Structures:

EFG was first applied by Krysl and Belytschko [12] to the analysis of thin plates for bending. The thin plate theory (or Kirchhoff plates) was employed and the boundary conditions were enforced by Lagrange multipliers. Gauss integration was performed on a background cell to evaluate the stiffness matrix. They have considered the plates of different dimensions and different loading conditions and the investigation was made for the effect of regular and irregular nodal distribution on the accuracy of solutions. Krysl and Belytschko has also conducted the analysis of thin shells using EFG method[14]. The geometrically exact theory of shear flexible shells was adopted and appropriate adjustments were made to account for the Kirchhoff–Love hypothesis. The membrane locking which appears in the numerical model was alleviated by enlarging the domains of influence of the EFG nodes for the quadratic basis, and it was removed completely by using quartic polynomial basis. Noguchi et al[34] extended EFG to the analysis of three-dimensional thin shell structures. The geometry of curved surface was expanded in a two dimensional space by using a mapping technique, and the nodes were generated on this two-dimensional mapped space. The bi-cubic and quartic basis functions were adopted for the construction of shape functions to eradicate shear and membrane locking. The alternative implementation of EFG method using a selective reduced integration formulation to remove volumetric locking in plates has been developed by Dolbow and Belytschko[25]. The shear locking was also investigated by Huerta and Fernan dez-Mendez[26], Kanok-Nukulchai et al[45]., and Askes et al [27].

Laminated composite plates and beams with piezoelectric patches using the EFG method were analyzed by Liew et al[50]. The first-order shear deformation plate theory was employed and the full transformations method was used to impose essential boundary conditions. The static shape control of piezo-laminated composite beams and plates was studied, and the influence of stacking sequence on the change in shapes was examined. The demonstrated numerical examples by EFG produces accurate solutions in analyzing the shape control of piezo laminated composite beams and plates. Peng et al.[99]carried out the bending analysis of un-stiffened and stiffened folded plates using the EFG method. More applications of the EFG method encompass the analyses of laminated folded plate structures [76], stiffened corrugated plates [72,81,84,97], functionally graded plates [69,91,103], and prestressed concrete beams [65] .

Bobaru and Mukherjee[47] presented a formulation for shape optimization of linear thermo elastic solids using the EFG method. They investigated the influence of the number of design parameters and observed that the EFG can give better results with a smaller number of design parameters than FEM. They also performed the shape sensitivity analysis and shape optimization in planner elasticity were also performed by the same author [41].

2.3.3 Buckling and Free Vibrations:

The elastic buckling behavior of stiffened and un-stiffened folded plates under partial in-plane edge loads was studied by Liew et al.[73] The formulation was based on the first-order shear deformation theory and element-free Galerkin method. The stiffness and initial stress matrices of the flat plates as determined by the mesh-free Galerkin method were superposed to obtain the stiffness and initial stress matrices of the entire folded plate. The solutions show the EFG method has a good accuracy and convergence rate. Liew et al. also conducted the buckling analysis of corrugated plates [74] and stiffened structures[79]. Zhao et al. [102] investigated the mechanical and thermal buckling response of functionally graded plates using an element-free kp-Ritz method. The first-order shear deformation plate theory was adopted to account for the transverse shear deformation and the shear locking was eliminated by using a stabilized nodal integration method. Also the effects of volume fraction exponent on the buckling response of functionally graded plates were examined.

Liu and Chen [39,57] studied the vibration response of the thin plates of complicated shape by using EFG method. Liu et al.[55] investigated the free vibration of thin shells structures. A free vibration analysis of folded plates was provided by Peng et al.[83] using the first-order shear deformation theory and the EFG method, and free vibration of sandwich beams with functionally graded core was investigated by Amirani et al.[101]

2.3.4 Non Linear Analysis:

In non linear analysis of structures Meshless methods have demonstrated great advantages and promising potential because they are flexible in handling discontinuities and large deformation problems, in which severe mesh distortion usually occurs when using FEM. Belinha and Dinis [82]conducted the non-linear analysis of laminated plates using element-free Galerkin method, and Ren et al[46]. performed the modelling and simulation of superelastic behaviours of shape memory alloys. The EFG method was also employed by other researcher for the non-linear analysis of folded plate structures[80], lower bound shakedown analysis of structures made of elastic-perfectly plastic material[96], non-linear wave propagation in damaged hysteretic materials[100], and simulation of and non-linear dynamic fracture[85]. In additional to the aforementioned meshless methods, other mesh-free methods that are based on EFG have been developed[43,70,71,75]. Additional application areas of meshless methods include transient analysis[52], probalistic mechanics and reliability[43], explicit dynamic analysis[32], linear and non-linear dynamic analysis of solids[68].

III. EFG Coupled With Other methods:

Element Free Galerkin (EFG) methods are methods for solving partial differential equations with the help of shape functions coming from Moving Least Squares Approximation. The EFG-method is more flexible than the Finite Element (FE) method, since it requires only nodal data and no element connectivity is needed. Because the EFG-method is computationally expensive, combinations of the EFG-method and the FE-method are considered.

For Moving Least Squares Approximation (MLSA) at each point of the domain the linear system should be solved. Moreover, when MLSA-shape functions are used in an implementation of a weak formulation. a dense integration pattern is necessary to get accurate values for the resulting linear system. Hence, the Element Free Galerkin (EFG) method is computationally expensive. Therefore, it is more convenient to make use of EFG only on a part of the domain where one wants to achieve an accurate approximation of the solution. And to make use of the Finite

Element (FE) method for the remaining part of the domain. [11]

3.1 Coupling Procedures of Meshfree Methods with FEM:

Considerable research in meshfree methods has been devoted on inherent difficulties like consistency, stability and Dirichlet boundary conditions. While these issues are not yet completely resolved, viable methods are available. In addition, up to now, the computational effort for meshfree methods is higher than for finite elements. Hence, as long as no robust and, at the same time, efficient formulation for meshfree particle methods is available, it seems beneficial to discretize only certain parts of the domain with particles and the rest with finite elements. [1,89,110]

Attaway et al[5]. has proposed one of the coupling procedure for meshfree methods and finite elements. They developed a master-slave coupling for fluid structure interactions; the fluid was discretized with particles, the structure was modelled with finite elements. Their algorithm is based on a common master slave coupling. A similar approach was proposed by Johnson and Johnson et al.[6] In addition, they developed transition elements where particles are fixed to FE nodes. This allows for a rigid coupling in the sense that tensile and shear forces are transferred through the interface.

Liu et al. [17] showed how to couple the reproducing kernel particle method (RKPM) with FEM by modifying the shape functions in the transition area for both RKPM and FEM. They applied the reproducing condition also in the transition area.

T. De Vuyst, R. Vignjevic and J.C. Campbell [67] developed a coupling algorithm for EFG and FEM by a mixed interpolation in the transition domain, where FE nodes are substituted by particles and connected via ramp functions to the EFG nodes so that continuity and consistency are preserved on the interface elements. They have also extended this method for a nodal integration of EFG. The only drawback of this method is, derivatives are discontinuous along the interface.

One more method for coupling EFG and FEM with Lagrange multipliers for elasto-static problems is presented by Hegen.[15] In his approach, the substitution of FE nodes by particles is not necessary. Rabczuk and Belytschko[64] extended this idea to nonlinear problems and applied it to deformable interfaces. Karutz [35] showed convergence of a similar approach to model crack propagation problems using an adaptively generated EFG domain.

Belytschko et al.[61] also invented a coupling with Lagrange multiplier, where finite elements and particles overlap. They called this method bridging domain coupling method and successfully applied it to atomic and multiscale simulations. Another bridging domain coupling was before proposed by Wagner and Liu [59] to couple atomic and continuum simulations. While the bridging domain coupling in suffered from spurious wave reflections in certain cases, Kadowaki and Liu [66] introduced some wave reflection algorithms to remove this drawback.

SPH-FE coupling by extending the SPH domain onto the FE mesh was proposed by Sauer[31]. In this method different possibilities for exchanging forces between finite element nodes and particles were shown. This approach was also used to convert elements into particles.

The main differences to most above mentioned methods is that they used a strong − form coupling.

Prof. W.K. Liu and S. Li [53]developed a hybrid method called reproducing kernel element method (RKEM) which exploits advantages of both, meshfree and finite element methods, e.g. the RKEM shape functions fulfill the Kronecker delta property. A method called moving particle finite element method was developed almost simultaneously by Hao et al.[61] While most hybrid FE-meshfree methods are at least first order in convergence, Liu et al. showed that their RKEM method maintains the usual convergence rate. Wagner and Liu and Han et al [54] have proposed Another method to maintain the usual convergence rate by hierarchical enrichment. The other good overviews about meshfree and particle methods, their coupling to finite elements with impressive examples can be found in Li and Liu. [62]

In the following articles the coupling procedure for EFG and FEM has been explained:

3.2 Compatibility coupling: Coupling with Lagrange multipliers:

The coupling approach where no ramp functions are needed was first developed by Hegen et al.[7] Rabczuk and Belytschko[64] used this approach to couple EFG nodes and finite elements to model the bond behavior in reinforced concrete beams in statics. In this method, relative displacements between the particles and the elements are allowed. The simplified version of the bridging domain coupling was developed by Xiao[49] named edge-to-edge coupling, which is an explicit method based on Lagrange multipliers. The main difference is that the FE and particle domain do not overlap which simplifies the method enormously. However, for wave propagation problems, the bridging domain coupling is more accurate. Coupling via lagrange multipliers has been reviewed in the following articles:

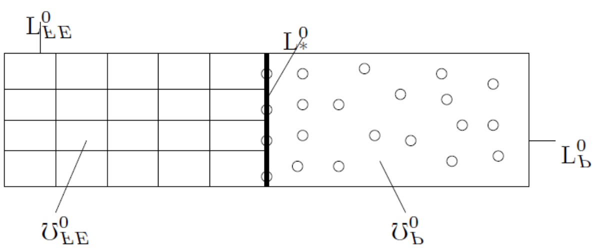

Figure3.1 : Coupling of Particle and Finite Elements [78]

For the static case, the potential to be minimized is

$$W = W^{int} - W^{ext} + \lambda^T g \tag{12}$$

Where W^int is the internal and W^ext is the external energy. The last term on the RHS is the constraints. In the dynamic case, an inertia term is added. The Lagrange multipliers are denoted by λ,and g=u^FE-u^p is the gap of the particle and the finite element domain along the common boundary as illustrated in figure The Lagrange multipliers are located at the particle positions and are:

$$g_h = \sum_{J=1}^{N} N_J^{FE}(X,t)u_J^{FE} - \sum_{J\in S} N_J^{P}(X,T)u_J^{P} \tag{13}$$

The Lagrange multiplier estimates are placed at the particle position and finite element shape functions are used to discretize the Lagrange multiplier field δλ:

$$\delta\lambda_h^P(X,t) = \sum_{J=1}^{N} N_J^{FE}(X,t)\delta\Lambda_J(t) \tag{14}$$

Note that for the interpolation in the above equation, the position of the Lagrange multipliers in the local element coordinate system has to be known. The test and trial functions are:

$$\delta u_h(X,t) = \sum_{J=1}^{N} N_J^{FE}(X,t)\delta u_J^{FE}(t) + \sum_{J\in S} N_J^{P}(X,t)\delta u_J^{P}(t) \tag{15}$$

$$u_h(X,t) = \sum_{J=1}^{N} N_J^{FE}(X,t)u_J^{FE}(t) + \sum_{J\in S} N_J^{P}(X,t)u_J^{P}(t) \tag{16}$$

And

$$N^{FE}(X,t) = 0\forall X \in \Omega_O^P \tag{17}$$

$$N^{P}(X,t) = 0\forall X \in \Omega_O^{FE} \tag{18}$$

Where S is the set of nodes in the particle model.

The derivatives of W^int and W^ext with respect to u are the internal and external forces, respectively. The additional forces λ ∂g/∂u are linear combinations of the Lagrange multipliers[64]. To obtain the discrete system of nonlinear equations Belytschko et al.[13] used the Taylor Expansion series for linearization. Substituting the test and trial functions, system equations can be obtained and matrices for force, stiffness also being obtained. The Lagrange multipliers are approximated to

$$\lambda_h^i = \sum_{J=1}^{N} \lambda_J \lambda_J^i \tag{19}$$

$$\delta\lambda_h^i = \sum_{J=1}^{N} \Lambda_J \delta\lambda_J^i \tag{20}$$

Where for the Λ the FE shape functions are chosen. Finally we obtain with the traction and displacement continuity the equation of motion.

$$\sum_{J=1} m_{IJ}\ddot{u}_J = f_I^{ext} - f_I^{int} \tag{21}$$

With

$$m = \begin{bmatrix} m^{FE} & 0 & -G^{FE} \\ 0 & m^P & G^P \\ -G^{FE} & G^P & 0 \end{bmatrix} \tag{22}$$

$$u = \begin{bmatrix} u^{FE} \\ u^P \\ \lambda \end{bmatrix} \tag{23}$$

$$F = \begin{bmatrix} f^{int,1} - f^{ext,1} \\ f^{int,2} - f^{ext,2} \\ -g \end{bmatrix} \tag{24}$$

And

$$m_{IJ}^i = \sum_I \int_{\Omega_0^i} \varrho_0^i N_I^i N_J^i \, d\Omega_0^i, \quad i = FE \text{ or } P \tag{25}$$

$$G_{IJ}^i = \int_{\Gamma_0^u} N_I^i \Lambda_J d\Gamma_0^i \tag{26}$$

$$f_I^{int,i} = \int_{\Omega_0^i} \nabla N_I^i P^i \, d\Omega_0^i, \quad i = FE \text{ or } P \tag{27}$$

$$f_I^{ext,i} = \int_{\Gamma_0^i} \varrho_0^i N_I^i b^i + N_I^I \bar{t}^i \, d\Gamma_0^i, \quad i = FE \text{ or } P \tag{28}$$

Where the superscript designates either the particle or FE domain. Note that we used here a consistent mass matrix.

We can couple other mesh free method also with FEM by using hydrodynamic coupling , master slave couplings, coupling via ramp functions, compatibility coupling, bridging domain coupling and hybrid coupling. Different coupling methods have some typical application, so they should be comparable on application bases. [18,27,28,30,33,40,56,60,78,106,107]

IV Conclusion

Mesh less methods and their applications in the analysis of engineering structures have been reviewed, with emphasis on the methods originating from EFG. The aim of this survey is to provide a general description of the developments and applications of Element Free Galerkin method. It has been demonstrated that the meshless methods are able to handle a variety of engineering problems, and offer great advantages over conventional numerical methods, especially in dealing with discontinuities and large deformation problems. However, there are still some challenges remaining. For three-dimensional modeling of structures, especially for thin shell structures, the computational cost is still too expensive. [1, 95,105,110]

The presented paper may consider as a quick view to the Element free galerkin- Mesh Free method, for the students who wanted to have research in this field. In that context one may find the complete overview of EFG, methodology, Application etc. regarding EFG method. The paper also gives some brief idea about combining Mesh free method with conventional FEM, to get the better use of both the methodology. [4, 6, 8, 9, 12, 22]

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